

Universal gates for transforming multipartite entangled Dicke states

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We determine the minimal number of qubits that are necessary to be accessed in order to transform Dicke states into other Dicke states. In general, the number of qubits in Dicke states cannot be increased via transformation gates by accessing only a single qubit, in direct contrast to other multipartite entangled states such as GHZ, W and cluster states. We construct a universal optimal gate which adds either spin-up qubits or spin-down qubits to any Dicke states by minimal access. We also show the existence of a universal gate which transforms any size of Dicke state as long as the gate has access to at least the required number of qubits for the transformation.

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I. INTRODUCTION

Entanglement is a key resource facilitating a wide range of emerging quantum technologies, such as quantum computing [1], communication [2, 3] and sensing [4]. It has been well established theoretically [5] and experimentally demonstrated between various particles, including photons, atoms and ions [6]. Entanglement between two particles [7] has been routinely prepared and used in different physical systems for a variety of tasks [1–5]. However, in order to make full use of the power of entanglement for quantum technologies and to probe deeper into the foundations of quantum mechanics, there has been an increasing push toward making larger numbers of particles entangled with each other. As the number of particles increases beyond two, different types of entangled states that cannot be converted into each other using local operations and classical communication (LOCC) [8] emerge. Greenberger-Horne-Zeilinger (GHZ) [9], cluster [10], Dicke [11] and W states [12], are examples of such inequivalent classes. This rich variety of structurally complex states among many particles holds great promise for a wide range of applications in quantum information. However, it is this same complexity that makes their preparation and manipulation difficult. Thus, understanding the limits for preparing and manipulating large multipartite entangled states are of great interest and urgently needed.

In this paper we derive the minimal number of qubits that are necessary to be accessed for expanding/reducing any given Dicke state. We show that, unlike W [13], GHZ and cluster states [14], Dicke states in general cannot be transformed by local access to only a single qubit. We consider gates for transforming Dicke states by minimal access. In the case of the expansion of W states, by accessing only one qubit there is a universal optimal gate which can expand any size of W states with maximum success probability. Similarly to such a case, we derive a universal optimal gate which adds either spin-up or spin-

down qubits to Dicke states by minimal access. We then construct universal gates which add/subtract given numbers of spin-up and spin-down qubits with a nonzero success probability, regardless of the size of an initial Dicke state. This work has important implications for assessing the amount of control one needs in the preparation and manipulation of Dicke states for quantum information applications, such as quantum algorithms [15], quantum games [16] and multi-agent quantum networking [17].

II. NECESSARY CONDITION FOR TRANSFORMING A DICKE STATE

An N -qubit Dicke state with M_1 excitations is the equally weighted superposition of all permutations of N -qubit product states with M_1 spin-up ($|1\rangle$) and $M_0 = N - M_1$ spin-down ($|0\rangle$), and is written as

$$|D_N^{M_1}\rangle = (C_N^{M_1})^{1/2} \hat{P} |M_0, M_1\rangle, \quad (1)$$

where $|M_0, M_1\rangle \equiv |M_0\rangle |M_1\rangle$ with $|M_i\rangle \equiv |i\rangle^{\otimes M_i}$ for $i = \{0, 1\}$, $C_N^{M_1} \equiv N!/(M_1!(N - M_1)!)$, and \hat{P} is a projector onto the symmetric subspace with respect to the permutation of any two particles. For example, $\hat{P}|2, 0\rangle = |00\rangle = |D_2^0\rangle$, $\hat{P}|1, 1\rangle = (|01\rangle + |10\rangle)/2 = \sqrt{2}|D_2^1\rangle$, and $\hat{P}|0, 2\rangle = |11\rangle = |D_2^2\rangle$.

We assume that $|D_N^{M_1}\rangle$ is shared between two subsystems A and B, and denote this as $|D_N^{M_1}\rangle_{AB}$, with subsystem A holding a total of k qubits and subsystem B holding the remaining qubits, as shown in Fig. 1 (a). Here we derive the minimum number k of qubits that should be accessed in order to transform the state $|D_N^{M_1}\rangle_{AB}$ into a state $|D_{N+n}^{M_1+m_1}\rangle_{AB}$, where $|n|$ is the total number of qubits added for $n > 0$ and deleted for $n < 0$, and similarly $|m_1|$ is the added/deleted number of qubits in $|1\rangle$, while $m_0 \equiv n - m_1$ represents for the added/deleted number of qubits in $|0\rangle$. For the trivial cases of $M_0 = 0$ ($M_0 + m_0 = 0$) and $M_1 = 0$ ($M_1 + m_1 = 0$), the states

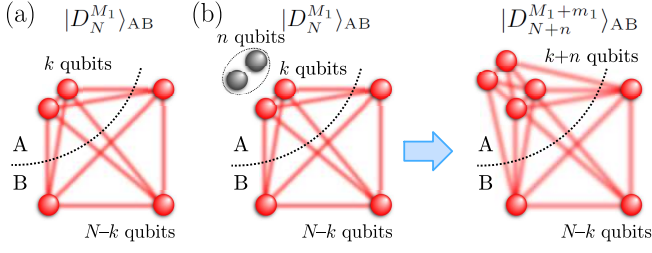


FIG. 1: (Color online) (a) The Dicke state $|D_N^{M_1}\rangle_{AB}$ shared between subsystems A and B, which hold k and $N-k$ qubits, respectively. (b) Expansion of $|D_N^{M_1}\rangle_{AB}$ to $|D_{N+n}^{M_1+m_1}\rangle_{AB}$ by accessing only k qubits in A. In the case of reduction, n qubits are eliminated from A.

of system AB are product states $|1 \dots 1\rangle$ and $|0 \dots 0\rangle$, respectively. In the following, we will study only the non-trivial cases where $M_0 > 0$, $M_1 > 0$, $M_0 + m_0 > 0$ and $M_1 + m_1 > 0$. We consider a local transformation scenario in which access to subsystem B is forbidden, and the transformation task is carried out by collectively manipulating the k qubits of subsystem A only [see Fig. 1 (b)]. In this scenario, the whole system after the transformation is composed of $N-k$ qubits in subsystem B and $k+n$ qubits in subsystem A. Thus, we have $N+n \geq N-k$, namely $N \geq k \geq -n$ is necessary.

We now derive a necessary condition for the transformation of Dicke states. The minimum number of spin-up qubits in subsystem B is given by $\alpha = \max\{M_1 - k, 0\}$ for the initial Dicke state $|D_N^{M_1}\rangle$, and is given by $\alpha' = \max\{M_1 + m_1 - k - n, 0\} = \max\{M_1 - k - m_0, 0\}$ for the final Dicke state $|D_{N+n}^{M_1+m_1}\rangle$. Since subsystem B is left untouched in the transformation, $\alpha' \geq \alpha$ should hold. Thus, $k \geq M_1$ is necessary for the transformation with $m_0 > 0$. Since a similar argument holds for the transformation with $m_1 > 0$, we have

$$k \geq \begin{cases} M_1 & \text{for } m_0 > 0, \\ M_0 & \text{for } m_1 > 0, \end{cases} \quad (2a)$$

$$(2b)$$

as a necessary condition for transforming a Dicke state to another Dicke state. Note that for other cases, we trivially have

$$k \geq -n \text{ for } m_0 \leq 0 \text{ and } m_1 \leq 0. \quad (3)$$

III. SUFFICIENT CONDITION FOR TRANSFORMING A DICKE STATE

Here we show that condition (2) is sufficient for transformation of a Dicke state to another Dicke state, and we derive the maximum probability for the transformation. We first decompose the Dicke state in Eq. (1) by using the symmetric bases in subsystems A and B. When we expand $C_N^{M_1} \hat{P}|M_0, M_1\rangle_{AB}$ in the computational basis, it is given by the sum of $C_N^{M_1}$ terms with unit amplitude. From these terms, select those that have j spin-up qubits in subsystem B. The sum of these selected terms should be given by $C_k^{M_1-j} \hat{P}|k - (M_1 - j), M_1 - j\rangle_A C_{N-k}^j \hat{P}|(N-k) - j, j\rangle_B$. Thus we can write $C_N^{M_1} \hat{P}|M_0, M_1\rangle_{AB} = \sum_{j=\alpha}^{\beta} C_k^{M_1-j} \hat{P}|k - (M_1 - j), M_1 - j\rangle_A C_{N-k}^j \hat{P}|(N-k) - j, j\rangle_B$, where the range of the summation over j is given by

$$\alpha = \max\{M_1 - k, 0\}, \quad \beta = \min\{N - k, M_1\}. \quad (4)$$

Using this decomposition, we rewrite Eq. (1) as

$$|D_N^{M_1}\rangle_{AB} = \sum_{j=\alpha}^{\beta} \sqrt{\frac{C_k^{M_1-j} C_{N-k}^j}{C_N^{M_1}}} |D_k^{M_1-j}\rangle_A |D_{N-k}^j\rangle_B. \quad (5)$$

For $k > -n$, decomposition of the desired state $|D_{N+n}^{M_1+m_1}\rangle_{AB}$ obtained from the transformation is similarly given by

$$|D_{N+n}^{M_1+m_1}\rangle_{AB} = \sum_{j=\alpha'}^{\beta'} \sqrt{\frac{C_{k+n}^{M_1+m_1-j} C_{N-k}^j}{C_{N+n}^{M_1+m_1}}} |D_{k+n}^{M_1+m_1-j}\rangle_A |D_{N-k}^j\rangle_B. \quad (6)$$

with $\alpha' = \max\{M_1 - k - m_0, 0\}$ and $\beta' = \min\{N - k, M_1 + m_1\}$. Since access is allowed only to subsystem A, the marginal state in subsystem B does not change through the transformation process, which implies the following relation as $\text{tr}_A(|D_N^{M_1}\rangle\langle D_N^{M_1}|) = p \text{tr}_A(|D_{N+n}^{M_1+m_1}\rangle\langle D_{N+n}^{M_1+m_1}|) + (1-p)\hat{\rho}_f^B$. Here p is the success probability of the transformation and $\hat{\rho}_f^B$ is the state of subsystem B when the transformation fails. From

Eq. (5), $\text{tr}_A(|D_N^{M_1}\rangle\langle D_N^{M_1}|)$ and $\text{tr}_A(|D_{N+n}^{M_1+m_1}\rangle\langle D_{N+n}^{M_1+m_1}|)$ are diagonalized by the basis $\{|D_{N-k}^j\rangle_B\}_{0 \leq j \leq N-k}$. Thus, from the positivity of $\hat{\rho}_f^B$, we have $p \leq p_{\max}$, where

$$p_{\max} \equiv \frac{C_{N+n}^{M_1+m_1}}{C_N^{M_1}} q_{\min} \quad (7)$$

with

$$q_{\min} \equiv \min_{\alpha' \leq j \leq \beta'} q_j \quad (8)$$

and

$$q_j \equiv \frac{C_k^{M_1-j}}{C_{k+n}^{M_1+m_1-j}}. \quad (9)$$

Here it should be understood that $C_0^0 \equiv 1$, and $C_k^{M_1-j} = 0$ for $M_1 - j < 0$ and $M_1 - j > k$.

When conditions (2a) and (2b) are satisfied, we have $\alpha \leq \alpha'$ and $\beta \geq \beta'$, respectively, resulting in $p_{\max} > 0$. Under the conditions, we construct a gate \mathcal{M}_A which achieves the upper bound on the success probability in Eq. (7). The gate \mathcal{M}_A is composed of a success operator \hat{M}_s and a failure operator \hat{M}_f satisfying $\hat{M}_s^\dagger \hat{M}_s + \hat{M}_f^\dagger \hat{M}_f = \hat{I}$. We define \hat{M}_s by

$$\hat{M}_s \equiv \sum_{j=\alpha'}^{\beta'} \sqrt{q_{\min} \frac{C_{k+n}^{M_1+m_1-j}}{C_k^{M_1-j}}} |D_{k+n}^{M_1+m_1-j}\rangle_{AA} \langle D_k^{M_1-j}|. \quad (10)$$

From Eq. (9), no coefficients of $\hat{M}_s^\dagger \hat{M}_s$ are larger than 1, and thus \mathcal{M}_A is a physically valid measurement process. From Eqs. (5), (6), (7) and (10), we have $\hat{M}_s |D_N^{M_1}\rangle_{AB} = \sqrt{p_{\max}} |D_{N+n}^{M_1+m_1}\rangle_{AB}$. As a result, the maximum probability of the transformation is p_{\max} defined in Eq. (7).

For $k = -n$, $m_0 \leq 0$ and $m_1 \leq 0$ are satisfied because $k \geq M_1 \geq -m_1 > -m_0 - m_1 = -n$ for $m_0 > 0$ and $k \geq M_0 \geq -m_0 > -m_0 - m_1 = -n$ for $m_1 > 0$. Thus the desired state after the transformation is $|D_{N-|n|}^{M_1-|m_1|}\rangle_B$. From the relation $\text{tr}_A(|D_N^{M_1}\rangle\langle D_N^{M_1}|) = p |D_{N-|n|}^{M_1-|m_1|}\rangle_{BB} \langle D_{N-|n|}^{M_1-|m_1|}| + (1-p) \hat{\rho}_f^B$ and the positivity of $\hat{\rho}_f^B$, we have $p \leq p_{\max}$, where p_{\max} is given by Eq. (7) with $\alpha' = \beta' = M_1 - |m_1|$ and $k = -n = |m_0| + |m_1|$, and is strictly positive. In such a case, the success operator for the gate \mathcal{M}_A which achieves the upper bound on the success probability is defined by

$$\hat{M}_s \equiv \sqrt{q_{\min}} |D_{|n|}^{|m_1|}\rangle, \quad (11)$$

which is a linear functional but we denote it as a linear operator for convenience here. From Eqs. (5) and (11), we obtain $\hat{M}_s |D_N^{M_1}\rangle_{AB} = \sqrt{p_{\max}} |D_{N-|n|}^{M_1-|m_1|}\rangle_B$. As a result, we conclude that conditions (2) and (3) are necessary and sufficient for the transformation of the Dicke states.

IV. UNIVERSAL OPTIMAL GATES FOR TRANSFORMING DICKE STATES BY ADDING ONE TYPE OF SPIN WITH MINIMAL ACCESS

When we expand a W state, which is a special case of Dicke states with only one excitation $M_1 = 1$, *i.e.*

$|W_N\rangle = |D_N^1\rangle$, a universal optimal gate \mathcal{M}_A which achieves the expansion to $|W_{N+m_0}\rangle = |D_{N+m_0}^1\rangle$ ($m_0 > 0$) by accessing only one qubit is constructed as $\hat{M}_s = |W_{n+1}\rangle_{AA} \langle 1| + \sqrt{(n+1)^{-1}} |0\rangle_A^{\otimes n+1} \langle 0|$. The expansion can be done regardless of the size of the initial W state as $\hat{M}_s |W_N\rangle = \sqrt{p} |W_{N+n}\rangle$ with success probability $p = (N+n)N^{-1}(n+1)^{-1}$, which coincides with p_{\max} calculated from Eqs. (7), (8) and (9) for any N .

Here we show that such a universal optimality is partially generalized to Dicke states under the following conditions: (a) the gate increases at most one type of spin, and (b) the gate accesses the minimum number of qubits to achieve the transformation.

For a gate with $m_0 \leq 0$ and $m_1 \leq 0$, the condition (b) means $k = -n = |m_0| + |m_1|$. Then the gate shown in Eq. (11) achieving p_{\max} only depends on m_1 and n . Thus it works as a universal optimal gate for any input with $M_0 \geq |m_0|$ and $M_1 \geq |m_1|$.

In the case of $m_0 > 0$, $m_1 \leq 0$, and $k = M_1$, we have $\alpha' = 0$, and q_j defined in Eq. (9) satisfies $q_j < q_{j+1}$ for $j = 0, 1, \dots, \beta' - 1$ [see Appendix]. As a result, we have $p_{\max} = q_0 C_{N+n}^{k-|m_1|} / C_N^k = C_{N+n}^{k-|m_1|} / (C_N^k C_{k+n}^{k-|m_1|})$ from definition (7). We give an explicit construction of a universal optimal gate \mathcal{M}_A^0 which transforms a Dicke state $|D_N^k\rangle$ to $|D_{N+m_0-|m_1|}^{k-|m_1|}\rangle$. The gate is characterized by the three parameters m_0 , m_1 and k , namely $\mathcal{M}_A^0 = \mathcal{M}_A^0(m_0, m_1, k)$. The gate \mathcal{M}_A^0 as a measurement process is represented by a success operator \hat{M}_{s_0} and a failure operator $\hat{M}_{f_0} = \sqrt{\hat{I} - \hat{M}_{s_0}^\dagger \hat{M}_{s_0}}$, and we define \hat{M}_{s_0} by

$$\hat{M}_{s_0} \equiv \sum_{j=0}^{k-|m_1|} \sqrt{\frac{C_{k+n}^{k-|m_1|-j}}{C_{k+n}^{k-|m_1|} C_k^{k-j}}} |D_{k+n}^{k-|m_1|-j}\rangle_{AA} \langle D_k^{k-j}|. \quad (12)$$

Since $\beta = \min\{M_0, k\}$ and $\beta' = \min\{M_0, k-|m_1|\}$, either $\beta \geq \beta' = k - |m_1|$ or $\beta = \beta' = M_0$ holds. Together with $\alpha = \alpha' = 0$, we see from Eqs. (5) and (12) that

$$\hat{M}_{s_0} |D_N^k\rangle_{AB} = \sqrt{\frac{C_{N+n}^{k-|m_1|}}{C_N^k C_{k+n}^{k-|m_1|}}} |D_{N+n}^{k-|m_1|}\rangle_{AB} \quad (13)$$

$$= \sqrt{p_{\max}} |D_{N+n}^{k-|m_1|}\rangle_{AB} \quad (14)$$

for any Dicke state $|D_N^k\rangle_{AB}$. Thus the gate \mathcal{M}_A^0 is the universal optimal gate for transforming Dicke states with minimal access of qubits for $m_0 > 0$ and $m_1 \leq 0$.

We can also construct a universal optimal gate for $m_1 > 0$, $m_0 \leq 0$ and $k = M_0$ by using the symmetry between $|0\rangle$ and $|1\rangle$. Let us define a new operator \hat{M}_{s_1} by interchanging the definition of $|0\rangle$ and $|1\rangle$ in Eq. (12), namely, replacing m_1 by m_0 and $|D_a^b\rangle_A$ by $|D_a^{a-b}\rangle_A$. After rewriting the parameter j by $k-j$, we arrive at

$$\hat{M}_{s_1} \equiv \sum_{j=|m_0|}^k \sqrt{\frac{C_{k+n}^{k+m_1-j}}{C_{k+n}^{m_1} C_k^{k-j}}} |D_{k+n}^{k+m_1-j}\rangle_{AA} \langle D_k^{k-j}|. \quad (15)$$

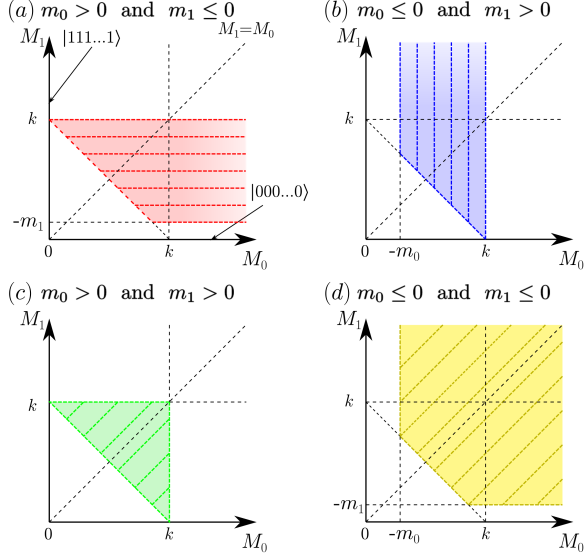


FIG. 2: (Color online) The successful operating areas of the universal gate $\mathcal{M}_A^{\text{univ}}$ defined in Eqs. (11) and (16) when (a) $m_0 > 0$ and $m_1 \leq 0$ are satisfied, (b) $m_0 \leq 0$ and $m_1 > 0$ are satisfied, (c) $m_0 > 0$ and $m_1 > 0$ are satisfied, and when (d) $m_0 \leq 0$ and $m_1 \leq 0$ are satisfied.

By symmetry, the gate \mathcal{M}_A^1 defined by \hat{M}_{s_1} and $\hat{M}_{f_1} = \sqrt{\hat{I} - \hat{M}_{s_1}^\dagger \hat{M}_{s_1}}$ achieves the optimal success probability p_{\max} when it is applied to $|D_N^{N-k}\rangle_{AB}$.

V. UNIVERSAL GATES FOR DICKE-STATE TRANSFORMATION

Here we derive universal gates $\mathcal{M}_A^{\text{univ}}(m_0, m_1, k) \equiv \{\hat{M}_s^{\text{univ}}, \hat{M}_f^{\text{univ}}\}$ which transform all Dicke states $|D_N^{M_1}\rangle_{AB}$ satisfying the conditions on M_0 and M_1 in Eqs. (2) and (3) to $|D_{N+n}^{M_1+m_1}\rangle_{AB}$ with nonzero success probabilities. For $k = -n = |m_0| + |m_1|$ with $m_0 \leq 0$ and $m_1 \leq 0$, it is easy to see that the gate defined in Eq. (11) is a universal gate for any Dicke state satisfying $M_0 \geq |m_0|$ and $M_1 \geq |m_1|$. In the following, we consider the case for $k > -n$.

We define the success operator of the gate by

$$\hat{M}_s^{\text{univ}} \equiv \sum_{j=\alpha_s}^{\beta_s} \sqrt{q_{\min}^k \frac{C_{k+n}^{k+m_1-j}}{C_k^{k-j}}} |D_{k+n}^{k+m_1-j}\rangle_{AA} \langle D_k^{k-j}|, \quad (16)$$

and define the failure operator by $\hat{M}_f^{\text{univ}} = \sqrt{\hat{I} - \hat{M}_s^{\text{univ}\dagger} \hat{M}_s^{\text{univ}}}$, where $\alpha_s \equiv \max\{0, -m_0\}$, $\beta_s \equiv \min\{k, k + m_1\}$, and

$$q_{\min}^k \equiv \min_{\alpha_s \leq j \leq \beta_s} \frac{C_k^{k-j}}{C_{k+n}^{k+m_1-j}} > 0. \quad (17)$$

Here the positivity comes from $\alpha_s \leq j \leq \beta_s$, implying that $0 \leq k - j \leq k$ and $0 \leq k + m_1 - j \leq k + n$. In Eq. (16), by substituting $j = k - M_1 + j'$, and relabelling j' as j , \hat{M}_s^{univ} is rewritten by

$$\hat{M}_s^{\text{univ}} = \sum_{j=\alpha''}^{\beta''} \sqrt{q_{\min}^k \frac{C_{k+n}^{M_1+m_1-j}}{C_k^{M_1-j}}} |D_{k+n}^{M_1+m_1-j}\rangle_{AA} \langle D_k^{M_1-j}|, \quad (18)$$

where $\alpha'' = \max\{M_1 - k, M_1 - k - m_0\}$ and $\beta'' = \min\{M_1, M_1 + m_1\}$. From Eqs. (10) and (18), \hat{M}_s^{univ} only differs from \hat{M}_s by the overall factor $\sqrt{q_{\min}^k/q_{\min}}$ and the range of the summation over j . Because $\beta = \min\{M_1, N - k\}$, $\beta' = \min\{N - k, M_1 + m_1\}$ and $\beta'' = \min\{M_1 + m_1, M_1\}$, either $\beta' = \beta''$ or $\beta = \min\{\beta', \beta''\}$ is satisfied. Similarly, because $\alpha = \max\{M_1 - k, 0\}$, $\alpha' = \max\{0, M_1 - k - m_0\}$ and $\alpha'' = \max\{M_1 - k - m_0, M_1 - k\}$, either $\alpha' = \alpha''$ or $\alpha = \max\{\alpha', \alpha''\}$ is satisfied. As a result, we have

$$\hat{M}_s^{\text{univ}} |D_N^{M_1}\rangle_{AB} = \sqrt{\frac{q_{\min}^k}{q_{\min}}} \hat{M}_s |D_N^{M_1}\rangle_{AB} \quad (19)$$

$$= \sqrt{\frac{q_{\min}^k}{q_{\min}} p_{\max}} |D_{N+n}^{M_1+m_1}\rangle_{AB}. \quad (20)$$

The success probability of the transformation is

$$p' = \frac{q_{\min}^k}{q_{\min}} p_{\max}. \quad (21)$$

From $q_{\min}^k > 0$, we see that the transformation succeeds with a nonzero probability whenever $p_{\max} > 0$.

For convenience, we classify the universal gates into four cases according to the signs of m_0 and m_1 , and show the range of applicable input Dicke states (M_0, M_1) for each class in Fig. 2. The input states outside of the designated region are not transformable by any means ($p_{\max} = 0$), while those in the area are transformed with a nonzero success probability by the gate $\mathcal{M}_A^{\text{univ}}$. Thus, the gates are universal gates for the Dicke-state transformation.

VI. CONCLUSION

Contrary to the expansion of GHZ, cluster and W states, Dicke states cannot be transformed by locally accessing only one qubit in general. We have derived the minimum number of qubits that should be accessed to transform a Dicke state to another Dicke state. Similarly to expansion of W states, when we access the minimum number of qubits for the transformation, one can construct universal optimal gates which add one type of spin to a given Dicke state. We have also constructed a universal optimal gate which deletes both types of spin from a given Dicke state with the minimum access of qubits. Finally, we have shown the existence of universal gates which transform any Dicke state satisfying the

derived condition for the transformation with nonzero probabilities. We believe that the results are important for understanding the amount of control needed in the preparation and manipulation of Dicke states for future quantum information applications.

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Appendix

Here we show that $q_j < q_{j+1}$ for $m_0 > 0$ and $m_1 \leq 0$. From Eq. (9), we have

$$q_j = \frac{k!}{(k+n)!} \frac{(j+m_0)!}{j!} \frac{(k-j-|m_1|)!}{(k-j)!}. \quad (22)$$

As a result, from $m_0 > 0$,

$$\frac{q_j}{q_{j+1}} = \frac{j+1}{j+1+m_0} \frac{k-j-|m_1|}{k-j} < 1, \quad (23)$$

namely $q_j < q_{j+1}$ is satisfied.

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